Problem 17

If the curve $y = e^{-x/10} \sin x$, $x \geq 0$, is rotated about the x-axis, the resulting solid looks like an infinite decreasing string of beads.

(a) Find the exact volume of the nth bead. (Use either a table of integrals or a computer algebra system.)

(b) Find the total volume of the beads.

Solution

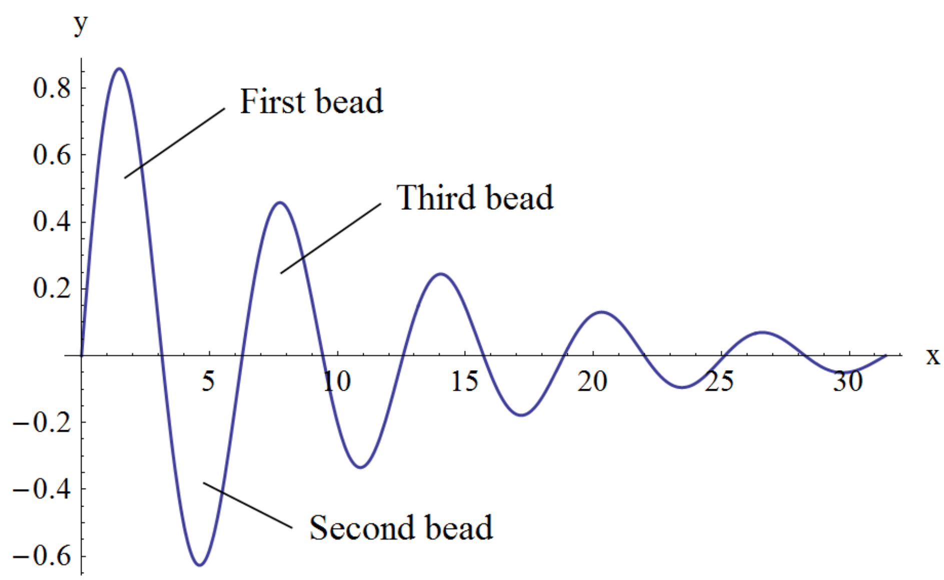

Figure 1: Plot of $y = e^{-x/10} \sin x$ for $0 < x < 10\pi$. Upon rotation about the x-axis, the areas under the curve will produce the beads.

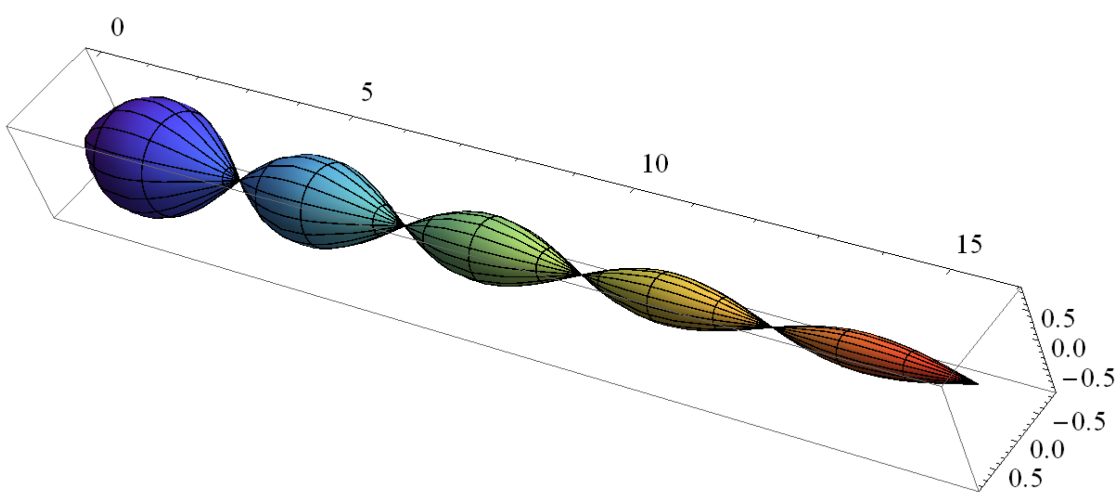

Figure 2: Rotating $y(x)$ about the x-axis results in this chain of beads for $0 < x < 5\pi$.

Part (a)

The volume of a bead is obtained by multiplying its cross-sectional area $A(x)$ by a little bit of thickness dx and integrating over the length of it.

$$V = \int_{\text{bead length}} A(x)\, dx$$

Because the volume is obtained by rotation about the x-axis, the cross-section is a circle with area $A(x) = \pi[y(x)]^2$, where $y(x)$ is the height of the given curve.

$$V = \int_{\text{bead length}} \pi[y(x)]^2\, dx$$

The first bead begins when $x = 0$ and ends when $x = \pi$, so its volume is given by

$$V_1 = \int_0^{\pi} \pi[y(x)]^2\, dx = \pi \int_0^{\pi} e^{-x/5} \sin^2 x\, dx = \frac{250\pi}{101} e^{-\pi/5}(e^{\pi/5} - 1) \approx 3.628.$$

The second bead begins when $x = \pi$ and ends when $x = 2\pi$, so its volume is given by

$$V_2 = \int_{\pi}^{2\pi} \pi[y(x)]^2\, dx = \pi \int_{\pi}^{2\pi} e^{-x/5} \sin^2 x\, dx = \frac{250\pi}{101} e^{-2\pi/5}(e^{\pi/5} - 1) \approx 1.935.$$

The third bead begins when $x = 2\pi$ and ends when $x = 3\pi$, so its volume is given by

$$V_3 = \int_{2\pi}^{3\pi} \pi[y(x)]^2\, dx = \pi \int_{2\pi}^{3\pi} e^{-x/5} \sin^2 x\, dx = \frac{250\pi}{101} e^{-3\pi/5}(e^{\pi/5} - 1) \approx 1.032.$$

Generalizing these results, we can obtain the volume of the nth bead.

$$V_n = \int_{(n-1)\pi}^{n\pi} \pi[y(x)]^2\, dx = \pi \int_{(n-1)\pi}^{n\pi} e^{-x/5} \sin^2 x\, dx = \frac{250\pi}{101} e^{-n\pi/5}(e^{\pi/5} - 1)$$

All of these integrals were calculated with Mathematica, a computer algebra system.

Part (b)

The total volume of the beads is obtained by summing V_n from $n = 1$ to ∞.

$$V = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} \frac{250\pi}{101} e^{-n\pi/5}(e^{\pi/5} - 1) = \frac{250\pi}{101}(e^{\pi/5} - 1) \sum_{n=1}^{\infty} e^{-n\pi/5}$$

This is a geometric series. The formula for evaluating one is given by

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r},$$

which converges, provided that $|r| < 1$. Thus,

$$\sum_{n=1}^{\infty} e^{-n\pi/5} = \sum_{n=1}^{\infty} (e^{-\pi/5})^n = \sum_{n=1}^{\infty} e^{-\pi/5} \cdot (e^{-\pi/5})^{n-1} = \frac{e^{-\pi/5}}{1 - e^{-\pi/5}} = \frac{1}{e^{\pi/5} - 1}.$$

And so

$$V = \frac{250\pi}{101}\cancel{(e^{\pi/5} - 1)} \cdot \frac{1}{\cancel{e^{\pi/5} - 1}}.$$

Therefore, the total volume of the beads is

$$V = \frac{250\pi}{101} \approx 7.776.$$